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Hierarchic Modeling of Plates

by

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Technical Note BN-1118

December 1990

INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Note BN-1118	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Hierarchic Modeling of Plates		5. TYPE OF REPORT & PERIOD COVERED Final life of contract
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) I. Babuska[1] and L. Li		8. CONTRACT OR GRANT NUMBER(s) [1]ONR N00014-90-J-1030
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Physical Science and Technology University of Maryland College Park, MD 20742		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE December 1990
		13. NUMBER OF PAGES 31
14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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Hierarchic Modeling of Plates

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MD-TR-90-54-BL

TR90-54

November 1990

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Abstract The paper surveys some of the results related to approach of hierarchic modeling of the plate problems. The main ideas are explained and illustrated by numerical examples.

1. Introduction

The problem of partial differential equations on a special domain ω, typically the "thin" domain is usually simplified by various dimensional reduction techniques. The aim is to approximately solve the 3 dimensional problem by a two dimensional formulation. This approach is widely applied in the connections *with plates and shells*. Today many plates and shells theories (models) are used in practice.

The principles of derivation of these models can be divided into 3 groups.

a) <u>Physical derivation</u>. Here various "a priori" assumptions of geometrical nature are made together with some additional assumptions involving importance of certain stress components. A typical example is the Kirchhoff hypothesis [1]. This approach was generalized by combination with various heuristic approaches to treat the anisotropic and laminated plates and shells. We mention [2], [3], [4] as an example of this type of derivation (see also references there).

b) <u>Asymptotic analysis</u> Here the main idea is to use the power series expansion (in the thickness) of the solution. It was analyzed with various levels of rigor. We mention here [5-12] and references mentioned there as typical for this approach.

Many results were devoted to the studies of the convergence (when the thickness $d \longrightarrow 0$) and the accuracy of the solution and the arguments why these models are relevant. We mention here for example [4], [11-16] and references there.

The approach of this type lead to the formulations of well known plates models, as for example, the Reissner-Mindlin model initiated in [8] [17] and which is a special case of the Naghdi plate theories [11].

Various plate models require various types of boundary conditions; more exactly some models allow to distinguish between certain boundary conditions, while others do not. As an example we mention that the Reissner-Mindlin model allows to distinguish between a soft and hard simple support, while the Kirchhoff model does not. These various boundary conditions could lead to very different properties of the solution. As an example we mention the paradox for *the hard simple support* [18-22] which says that the solution of the hard simple support on a regular n-gone inscribed in the unit circle does not converge to the solutions on the circular domain as $n \longrightarrow \infty$. In contrast, *for the soft simple* support, this paradox is not present. The hard support paradox also occurs in the 3 dimensional theory [21].

The properties of solution of the Reissner-Mindlin model especially its boundary layer behavior for smooth domain was studied rigorously in [23], [24]. In [25] the boundary layer for unsmooth domain was heuristically analyzed. The behavior of the solution in the neighborhood of the corners of the domain was analyzed in [26] [27]. For the survey of various theories for laminated plates we refer also to [4].

c) <u>Numerical, hierarchical approaches</u>. This approach is a modern one, and typical for the computer oriented procedures. It is stemming from general principles of numerical methods, adaptive approaches and a-posteriori error estimations. Here no particular model is a-priori preferred. This approach constructs adaptively an optimal numerical method for solving the three dimensional problem which leads to the accurate computation of data of interest.

In the next sections we will elaborate on this approach. For the sake of concreteness we will address these questions on the specific model problems although the results hold in a general setting.

2

2) The three dimensional formulation of the plate problem

Let $\omega \in \mathbb{R}^2$ be a bounded domain with the boundary Γ and $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 | (x_1, x_2) \in \omega, \ -\frac{d}{2} < x_3 \leq \frac{d}{2}\}$. Further we let

$$S = \left\{ x \in \mathbb{R}^3 | (x_1, x_2) \in \Gamma, \ -\frac{d}{2} < x_3 < \frac{d}{2} \right\}$$

$$R_\pm = \left\{ x \in \mathbb{R}^3 | (x_1, x_2) \in \omega, \quad x_3 = \pm d \right\}.$$

We will refer to the solution of *the three dimensional elasticity problem* for an isotropic homogeneous material when the equal normal load $\frac{1}{2} q(x)$ is acting on $R_\pm$, as the exact solution of the plate problem. The solution $u = (u_1, u_2, u_3)$ is *the minimizer of the total energy*

$$(2.1) \qquad\qquad G^A(u) = \mathcal{E}^A(u) - Q(u)$$

over the set of functions $\mathcal{K}(\Omega) \subset (H^1(\Omega))^3$ which satisfy certain constraints on S. The boundary conditions of the plate problems are uniquely characterized by this set $\mathcal{K}(\Omega)$.

In (2.1) we denoted by $\mathcal{E}^A(u)$, the strain energy based on the Hooke's compliance matrix A for the plate material.

The problem has been formulated for simplicity only for the homogeneous boundary conditions. For concreteness and simplicity we will deal with the square plate which we denote by Ω_s resp. $\omega_s = \{x_1, x_2 | \ |x_i| < 0.5, \ i = 1, 2\}$ and

$$\Gamma_1 = \{x_1, x_2 | \ x_1 = 0.5, \ |x_2| < 0.5\}$$

$$\Gamma_2 = \{x_1, x_2 | \ x_1 < 0.5, \ x_2 = 0.5\}$$

$$\Gamma_3 = \{x_1, x_2 | \ x_1 = -0.5, \ |x_2| < 0.5\}$$

$$\Gamma_4 = \{x_1, x_2 | \ |x_1| < 0.5, \ x_2 = -0.5\}$$

$$S_i = \Gamma_i \times (-d/2, \ d/2), \ i = 1, 2, \cdots, 4.$$

The constrain $\mathcal{K}(\Omega)$ defines various boundary conditions. Let us show some typical ones which could be imposed for example on $S_1 = \Gamma_1 \times (-d/2, d/2)$:

a) clamped $\qquad\qquad\qquad\qquad\quad u_1 = u_2 = u_3 = 0$ on S_1

b) free $\qquad\qquad\qquad\qquad\qquad\quad$ no constrain on S_1

c) hard simple support: $\qquad\quad u_2 = u_3 = 0$ on S_1

d) soft simple support: $\qquad\quad u_3 = 0$ on S_1

e) very soft simple support: $\displaystyle\int_{-d/2}^{d/2} u_3(x_1,x_2,x_3)\, dx_3 = 0,\quad (x_1,x_2) \in \Gamma_1$

f) illegal simple support: $\quad u_3(x_1,x_2,0) = 0,\quad (x_1,x_2) \in \Gamma_1$.

The last type of the simple support is called "illegal", because its solution is the same as for the free boundary condition. Mathematically, this follows from the fact that functions of $(H^1(\Omega))^3$ do not have trace on the one dimensional line $\hat{\Gamma}_1 = \{x_1,x_2,x_3 \mid x_1,x_2 \in \Gamma_1 \; x_3 = 0\}$.

Physically it follows from the observation that the displacement under load concentrated on a line is infinite (Bousinesque solution).

The solution of the three dimensional problem with finite strain energy exists and is uniquely determined (except of the case of free boundary conditions on Γ when the usual additional conditions have to be imposed).

The aims of the computation can be different. For example the interest can be in the bending moments and shear forces or average displacements, etc. Further the interest could be only in these values inside the domain ω or at the boundary Γ or in the neighborhood of Γ. We can be interested in the stresses or in the stress intensity functions along the edges and stress intensity factors in the vertices etc.

It is obvious that any imposed boundary condition is an idealization. Hence one can study how the results are influenced by the uncertainty of the idealization of the boundary conditions. We will not address this problem in details here. Instead we show an example of the modeling of the clamped in boundary condition for the plate in the case when the solution is independent of the variable x_2.

Let us consider the plate of thickness $d = 1$ and span $L = 20$ with the modeled clamped end as shown in Fig. 2.1. We will assume that the material is homogeneous, its modulus of elasticity $E = 3.10^7$ and Poisson ratio $\nu = 0.3$. Boundary condition e) is the spring condition with the spring constant $C = 10^8$. Case f in the Fig. 2.1 is the strength of material model based on the Kirchhoff theory. In this case the bending moments in the middle of the plate is $M = \frac{1}{24} 20^2 q = \frac{100}{6} q$. Normalizing the results by the moment of the model d) we give the error in the bending moment M in the middle of the plate in the Table 2.1.
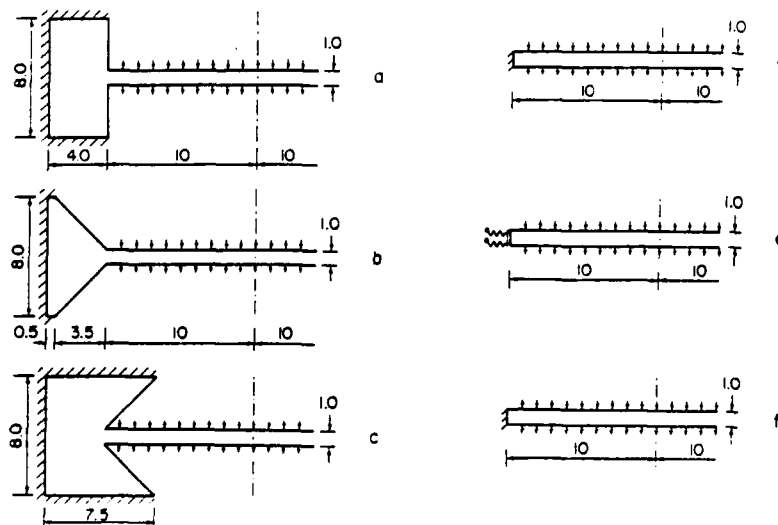

Figure 2.1   The scheme of various boundary conditions

Table 2.1   The error of various models of boundary conditions

Case	Error in M
a	+ 8.2%
b	+ 9.5%
c	+ 8.0%
d	0%
e	+19.9%
f	- 0.2%

Table 2.1 clearly shows the influence of the uncertainties in the boundary condition on the bendings moment. The influence on the stresses in the neighborhood of the boundary condition is much larger. For more see [28].

3. The principles of the numerical hierarchical modeling

Let us assume the following form of the displacement

$$(3.1) \qquad u_i(x_1,x_2,x_3) = \sum_{j=0}^{n_i} u_{i,j}(x_1,x_2)\ \varphi_{i,j}\!\left(\frac{2x_3}{d}\right), \quad i = 1,2,3$$

where the functions $\varphi_{i,j}(\eta)$, $-1 < \eta < 1$, may in general depend on d and $n = (n_1\ n_2,n_3)$, $n_i \geq 0$, $i = 1,2,3$ are integers. Because of our assumptions about the symmetry of the loading we can assume that $\varphi_{i,j}(\eta)$ are anti-symmetric functions for $i = 1,2$ and symmetric for $j = 3$. Then $u_{i,j} = 0$ for $i = 1,2$ and j even and $u_{3,j} = 0$ for j odd. We will assume that $u \in (H^1(\Omega))^3 \cap \mathcal{K}(\Omega)$ and n can be different in different regions of ω.

The following questions arise

a) how to select the functions $\varphi_{i,j}(\eta)$

b) how to select n; uniform or with different values in different regions of ω

6

c) how to derive the differential equations or variational formulations to determine the functions $u_{i,j}$.

d) how to assess the accuracy of the solution with respect to the data of interest (when compared with the data of exact 3 dimensional solution), how to design an adaptive procedure.

Specifying the functions $\varphi_{i,j}(\eta)$ and the principles of derivation of the differential equations for a sequence $n^{(i)} = (n_1^{(i)}, n_2^{(i)}, n_3^{(i)})$, $i = 1,2$, and any thickness d *we construct a family of n-models*. This family will be called *hierarchical* if it satisfies the following conditions.

i) Let u_d and u_{nd} be the solution of the 3 dimensional problem of the plate of thickness d and the solution of the n-model respectively. Then

$$(3.2) \qquad \| u_d - u_{nd} \| / \| u_d \| \longrightarrow 0 \quad \text{as} \quad d \longrightarrow 0$$

holds.

Mostly but not exclusively the energy norm $\| \cdot \|$ in (3.2) is considered.

ii) if the solution u_d is sufficiently smooth uniformly with respect to d then

$$(3.3) \qquad \| u_d - u_{n_i d} \| / \| u_d \| \leq C(n_i) d^{\alpha(n_i)}$$

where $\alpha(n_{i+1}) > \alpha(n_i)$

iii) for any fixed d we have

$$(3.4) \qquad \| u_d - u_{n_i d} \| / \| u_d \| \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty$$

and preferably but not necessarily $\| u_d - u_{n_i d} \| \geq \| u - u_{n_{i+1} d} \|$.

We remark that the notion of hierarchy depends strongly on the norm $\| \cdot \|$. We can also define the hierarchic family by replacing the norm $\| \cdot \|$ by another definition of accuracy, which expresses the goal of the analysis.

4. The basic hierarchy

Let us assume that functions $\varphi_{i,j}(\eta)$ in (3.1) are *independent of* d. Then in the spirit of the requirement (3.3), we can ask which $\varphi_{i,j}$'s lead to that highest α. The answer is: $\varphi_{i,j}$ *have to be polynomial of degree* j. For the arguments leading to this conclusion we refer to [29], [30], [31]. Hence (3.1) can be written as

$$(4.1) \qquad u_i(x_1,x_2,x_3) = \sum_{j=0}^{n_i} u_{i,j}(x_1,x_2) \left[\frac{2x_3}{d}\right]^j, \quad i = 1,2,3$$

and as before $u_{i,j} = 0$ for $i = 1,2$ and j even and $u_{3,j} = 0$ for j odd. The main models we can therefore consider are: $n = (1,1,0)$, $(1,1,2)$, $(3,3,2)$, $(3,3,4)$ etc.

Let us now address the question of deriving the differential equation for $u_{i,j}$. We define $u_{i,j}$ as the *minimizers of the potential energy*

$$(4.2) \qquad G^B(u) = \mathcal{E}^B(u) - Q(u)$$

over all u of the form (4.1) belonging to $\mathcal{K}(\Omega) \subset (H^1(\Omega))^3$ (see also (2.1)) and B being a Hooke's compliance matrix *which has not to be* identical with the matrix A of the plate material. We have

$$(4.2) \quad A = \begin{bmatrix} \dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \dfrac{\nu E}{(1+\nu)(1-2\nu)} & \dfrac{\nu E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\[2ex] & \dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \dfrac{\nu E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\[2ex] & & \dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\[2ex] \text{Symmetric} & & & \dfrac{E}{1+\nu} & 0 & 0 \\[2ex] & & & & \dfrac{E}{1+\nu} & 0 \\[2ex] & & & & & \dfrac{E}{1+\nu} \end{bmatrix}$$

As usual, we denote by E, the modulus of elasticity, and by ν, the Poisson ratio.

We have

__Theorem 4.1__ The model $(1,1,0)$ with

$$u_1 = u_{1,1} \frac{2x_3}{d}$$

(4.4)
$$u_2 = u_{2,1} \frac{2x_3}{d}$$

$$u_3 = u_{3,0} \quad \text{and}$$

$$B = A$$

is for $\nu \neq 0$ not the member of the hierarchic family (it violates condition (3.2)).

For more see [32], [33].

Let

(4.5)
$$R = \begin{bmatrix} \dfrac{E}{1-\nu^2} & \dfrac{\nu E}{1-\nu^2} & 0 & 0 & 0 & 0 \\[2mm] & \dfrac{E}{1-\nu^2} & 0 & 0 & 0 & 0 \\[2mm] & & \dfrac{E}{1-\nu^2} & 0 & 0 & 0 \\[2mm] \text{Symmetric} & & & \dfrac{E}{1+\nu} & 0 & 0 \\[2mm] & & & & \dfrac{E\kappa}{1+\nu} & 0 \\[2mm] & & & & & \dfrac{E\kappa}{1+\nu} \end{bmatrix}$$

where $0 < \kappa < 1$ arbitrary. Then we have

__Theorem 4.2__ The model $(1,1,0)$ is the member of the hierarchic family when the matrix $B = R$ defined by (4.5) is used. Then for smooth solution $\|u_d - u_{nd}\|/\|u_d\| \leq Cd$ i.e. $\alpha = 1$. Model $(1,1,0)$ with matrix $B = R$ is the well known Reissner-Mindlin model (see [32], [33]).

The coefficient κ in (4.5) is the shear factor with different values recommended in the literature. The typical recommendation is $\kappa = 5/6$ or $0.833 < \kappa < 0.870$ as recommended in [34].

Let us consider the example of square plate Ω_s (see Section 2.1) uniformly loaded with simple soft support, and $d = 0.01$. Assume $E = 10^7$ and $\nu = 0.3$. Further by $\xi(\kappa)$ we denote

$$(4.6) \qquad \xi(\kappa) = \left[\frac{|\mathcal{E}(\kappa) - \mathcal{E}(3 \text{ dim})|}{|\mathcal{E}(3 \text{ dim})|} \right]^{1/2}$$

where by $\mathcal{E}(3 \text{ dim})$ we denoted the strain energy of the 3 dimensional solution and by $\mathcal{E}(\kappa)$ the strain energy of the $(1,1,0)$ model with $B = R$ given by (4.5).

In the Table 4.1 we show the energy $\mathcal{E}(\kappa)$, $\mathcal{E}(3 \text{ dim})$ for various κ and the error $\xi(\kappa)$.

Table 4.1 Energy $\mathcal{E}(\kappa)$ as function of κ and the error ξ.

κ	$\mathcal{E}(\kappa) \times 10^3$	$\xi\%$
1	0.234563	2.69
0.91	0.234674	2.26
0.87	0.234729	0.41
0.8453	0.234765	1.16
5/6	0.234783	1.46
KIRCHHOFF	0.232392	9.98
3 DIM	0.234733	--

In the Table 4.1 we also reported the solution of Kirchhoff model. We see that $\kappa = 0.87$ leads here to the best result and the Kirchhoff model has a large error in the energy norm.

Remark The solution of the 3 dimensional formulation is not known in the analytic form. This solution was numerically computed by the hp-version of

10

FEM with refined meshes and error estimation which guarantee that the data reported here are adequately accurate, i.e., the error does not influence our conclusions.

As an illustration of Theorem 4.2 consider the problem of clamped-in plate, uniformly loaded. In the table 4.2 we show $\xi(0.87)$ as function of d

Table 4.2. The energy $\mathcal{E}(\kappa)$ and the error ξ as function of d.

d	3 DIM	RM, $\kappa=0.87$	$\xi_{RM\%}$
0.10	0.242115(-6)	0.245521(-6)	11.8%
0.025	0.149115(-4)	0.149256(-4)	3.07%
0.01	0.232489(-3)	0.232524(-3)	1.2%

We clearly see the convergence rate $\alpha = 1$.

The shear factor influences only the two terms $\frac{E}{1+\nu}$ in the lower right corner of the matrix A. Hence we could consider also for higher models the matrix A^* with the shear factor influencing these two values.

We have

<u>Theorem 4.3</u> The models $(n,n,n+1)$, $n = 1,2,\cdots$, using the matrix $B = R$ do not belong to the hierarchic family because condition (3.2) does not hold.

<u>Theorem 4.4</u> The sequence of the models $(n,n,n+1)$, $n = 1,2,\cdots$ is hierarchic for $B = A$ and is not hierarchic when $B = A^*$.

Table 4.3 which addresses the problem of the square plate with soft support illustrates Theorem 4.4.

Table 4.3   Energy $\mathcal{E}$ for different models and the error ξ

κ	Model	$\mathcal{E} \cdot 10^3$	$\xi\%$	B
1	(1,1,2)	0.234528	2.95	A
1	(3,3,4)	0.234731	0.29	A
1	(5,5,6)	0.234732	0.01	A
0.87	(1,1,2)	0.234693	1.3	A*
0.87	(3,3,4)	0.234913	2.76	A*
0.87	(5,5,6)	0.234914	2.78	A*
	3 dim	0.234733		

Remark In all examples we mention the exact solutions of the differential equations of the models. They were solved numerically with the error control so that the reported data are exact in the range we used for our conclusions.

5)   Capturing the boundary layer

As has been said above, we compare the solution of the n-model with the solution of the three dimensional problem. The accuracy then has to be assessed relatively to the data of interest.

Let us be interested in the behavior and accuracy of the bending moments and shear forces in the neighborhood of the boundary.

Consider the problem of unit square plate Ω_s (d = 0.01) and compare the values of the twist moment $M_{12}(0.4, x_2)$ computed from the 3 dimensional formulation, RM model (i.e. (1,1,0) model with B = R and κ = 0.87) and (1,1,2) model with B = A, κ = 1.

Table 5.1a Twist moment $M_{12}(0.4, x_2)$ for the hard and soft simple support.

x_2	HARD			SOFT		
	3 DIM	RM	(1,1,2)	3 DIM	RM	(1,1,2)
0	0	0	0	0	0	0
0.02368	0.002	0.002	0.002	0.002	0.001	0.002
0.11842	0.009	0.009	0.009	0.009	0.009	0.009
0.21316	0.016	0.016	0.016	0.016	0.016	0.016
0.45000	0.029	0.029	0.029	0.029	0.029	0.029
0.48079	0.029	0.029	0.029	0.030	0.030	0.030
0.49026	0.029	0.029	0.029	0.029	0.029	0.028
0.49500	0.029	0.029	0.029	0.024	0.024	0,025
0.49713	0.030	0.030	0.039	0.018	0.018	0.019
0.49903	0.030	0.030	0.030	0.008	0.008	0.008
0.49950	0.030	0.030	0.030	0.005	0.005	0.005
0.50000	0.030	0.030	0.030	0	0	0

Table 5.1b Twist moment $M_{12}(0.5, x_2)$ for hard support.

	3 DIM	RM	(1,1,2)
0	0	0	0
0.02368	0.002	0.002	0.002
0.11842	0.010	0.010	0.010
0.21316	0.017	0.017	0.017
0.45000	0.031	0.031	0.031
0.48079	0.032	0.032	0.032
0.49026	0.032	0.032	0.032
0.49500	0.032	0.032	0.032
0.49713	0.033	0.033	0.033
0.49903	0.033	0.033	0.033
0.49950	0.033	0.033	0.033
0.50000	0.035	0.035	0.035

We see clearly that the boundary layer is present for the soft support and is practically not existent for the hard support. Let us remark that $M_{12}(0.5, x_2) = 0$ for the soft support.

In the tables 5.2a, b we show the analogous results for the shear forces Q_{31} and Q_{32}.

Table 5.2a. The shear forces $Q_{31}(0.4, x_2)$ for the hard and soft support

x_2	HARD			SOFT		
	3 DIM	RM	(1,1,2)	3 DIM	RM	(1,1,2)
0	0.246	0.246	0.246	0.245	0.246	0.246
0.02368	0.245	0.245	0.245	0.245	0.245	0.245
0.11842	0.235	0.234	0.234	0.235	0.234	0.234
0.21316	0.208	0.208	0.208	0.208	0.208	0.208
0.45000	0.052	0.053	0.053	0.053	0.056	0.055
0.48079	0.020	0.021	0.021	-0.005	0.001	0.006
0.49026	0.010	0.011	0.011	-0.419	-0.403	-0.343
0.49500	0.005	0.005	0.005	-1.921	-1.912	-1.822
0.49713	0.003	0.003	0.003	-3.775	-3.817	-3.824
0.49903	0.001	0.001	0.001	-6.974	-7.047	-7.379
0.49950	0.001	0.000	0.000	-8.167	-8.214	-8,695
0.50000	0	0.000	0.000	-9.718	-9.639	-10.323

Table 5.2b The shear force $Q_{3i}(0.5, x_2)$ and $Q_{32}(0.5, x_2)$ for the soft support

x_2	$Q_{31}(0.5, x_2)$			$Q_{32}(0.5, x_2)$		
	3 DIM	RM	(1,1,2)	3 DIM	RM	(1,1,2)
0	0.420	0.420	0.420	0	0	0
0.02368	0.420	0.420	0.420	-0.643	-63801	-0.685
0.11842	0.408	0.408	0.408	-3.199	-3.173	-3.399
0.21316	0.374	0.374	0.374	-5.663	-5.617	-6.018
0.45000	0.133	0.133	0.135	-10.451	-10.363	-11.005
0.48079	-0.133	-0.125	-0.105	-10.635	-10.559	-11.332
0.49026	-0.813	-0.807	-0.756	-9.905	-9.844	-10.651
0.49500	-2.175	-2.188	-2.157	-8.600	-8.505	-9.251
0.49713	-3.514	-3.573	-3.645	-7.706	-7.569	-8.221
0.49903	-5.462	-5.499	-5.815	-6.954	-6.881	-7.368
0.49950	-6.069	-6.073	-6.485	-6.838	-6.727	-7.252
0.50000	-6.793	-6.678	-7.221	-6.793	-6.678	-7.221

Let us analyse now the strength of the boundary layer. To this end let us define the functions $\beta_{31}(x_1, x_2)$.

$$\exp\left[-\beta_{31}(x_1, x_2) \ (0.5 - x_2)/d\right]$$

$$= \left|\frac{Q_{31}^H(x_1, x_2) - Q_{31}^S(x_1, x_2)}{Q_{31}^S(x_1, 0.5)}\right|$$

Because the hard support solution has no boundary layer, we use it as a "base" smooth function. Functions β_{31} characterizes the strength of the boundary layer. Table 5.3 shows the values of $\beta_{31}(x_1, x_2)$ for $x_1 = 0.4, 0.5$.

Table 5.3 The strength function $\beta_{31}(x_1, x_2)$ of the boundary layer for the soft support

x_2	$x_1 = 0.4$			$x_1 = 0.5$		
	3 DIM	RM	(1,1,2)	3 DIM	RM	(1,1,2)
0.49263	3.22	3.25	3.43	2.17	2.21	2 37
0.49500	3.24	3.24	3.47	2.27	2.23	2.41
0.49666	3.28	3.24	3.47	2.28	2.20	2.39
0.49878	3.40	3.23	3.47	2.23	2.04	2.24
0.49950	3.51	3.23	3.47	2.22	1.91	2.12
0.49989	3.62	3.24	3.47	2.22	1.83	2.12
0.49991	3.63	3.24	3.47	2.21	1.83	2.03
0.49994	3.63	3.23	3.47	2.21	1.83	2.03

It has been shown in [23] [24] that for the smooth boundary $\beta_{31}(x_1, x_2)$ = $\sqrt{12\kappa}$ = 3.23 (in our case) which can be used if $x_1 \approx 0.5$. In [25] is suggested that $\beta_{31}(0.5, x_2) \approx \sqrt{6\kappa}$ = 2.28. We see that the value 3.23 very well describes the character of the boundary layer while the exact theoretical strength for $x_1 = 0.5$ is yet an open question. We also see that the boundary layer of the 3 dimensional solution as well as of the model (1,1,2) is stronger than of the RM model. We will return to this question later in this section.

As $d \longrightarrow 0$ then the difference between hard and smooth support disappears when measured in the energy norm. It is not the case for the data at the boundary. We can expect for example that $dQ_{31}(0.5, 0.5)$ converges as $d \longrightarrow 0$ to some value (although not yet proven).

In the table 5.4 we show the values of $Q_{31}(0.5, 0.5)$ and $dQ_{31}(0.5, 0.5)$ computed from RM model for $\kappa = 5/6$.

Table 5.4 The values of $Q_{31}(0.5, 0.5)$ for RM model as function of d for the soft simple support.

d	$Q_{31}(0.5, 0.5)$	$dQ_{31}(0.5, 0.5)$
0.025	-2.68	-0.0671
0.01	-6.54	-0.0654

We can expect that for fixed d, $Q_{31}(0.5, 0.5)$ is proportional to $\sqrt{\kappa}$. In Table 5.5 we show some values for $d = 0.01$.

Table 5.5 $Q_{31}(0.5, 0.5)$ as function of κ for the RM model

κ	$Q_{31}(0.5, 0.5)$	$\kappa^{-1/2} Q_{31}(0.5, 0.5)$
1	-7.15	-7.15
0.91	-6.83	-7.16
0.87	-6.68	-7.16
5/6	-6.54	-7.16

So far we have seen that the RM model describes reasonably well the moments although the strength of the boundary layer is more than 10% off.

Our discussion was for the soft simple support. For other boundary conditions the situation can be different. As an example we consider the problem of the square plate Ω_s of thickness $d = 0.01$ with all four sides clamped-in. In the Table 5.6 we report the values of the moments $M_{11}(x_1, 0)$ and $M_{22}(x_1, 0)$.

Table 5.6 The moment $M_{11}(x_1,0)$ and $M_{22}(x_1,0)$ for various models

x_1	$M_{11}(x_1,0)$			$M_{22}(x_1,0)$		
	3 DIM	RM	(1,1,2)	3 DIM	RM	(1,1,2)
0	-0.0229	-0.0229	-0.0229	-0.0229	-0.0229	-0.0229
0.0000	-0.0157	-0.0157	-0.0157	-0.0157	-0.0163	-0.0163
0.4000	+0.0164	+0.0163	+0.0164	+0.0026	+0.0027	+0.0026
0.4900	0.0470	0.0470	0.0470	0.0141	0.0141	0.0141
0.4930	0.0483	0.0483	0.0483	0.0145	0.0144	0.0144
0.4990	0.0509	0.0509	0.0509	0.0168	0.0152	0.0170
0.4993	0.0510	0.0510	0.0510	0.0176	0.0153	0.0179
0.4999	0.0512	0.0512	0.0512	0.0207	0.0153	0,0212
0.5000	0.0513	0.0513	0.0513	0.0220	0.0153	0.0220

We see that the error in $M_{11}(x_1,0)$ is negligible for RM and the (1,1,2) model and no boundary layer is present. The moment $M_{22}(x_1,0)$ computed from the RM model has no boundary layer either, but has error 30%. In contrast, 3 dimensional solution shows very strong boundary layer.

We will analyze the strength of the boundary layer as before. Let us define $\beta_{22}(x_1,x_2)$

$$\exp\left[-\beta_{22}(x_1,x_2)\left[\frac{(0.5-x_2)}{d}\right]\right] =$$

$$\left|\frac{M_{22}^{RM}(x_1,x_2) - M_{22}(x_1,x_2)}{M_{22}^{RM}(0.5,x_2) - M_{22}(0.5,x_2)}\right| .$$

Here we did use $M_{22}^{RM}(x_1,x_2)$ as the smooth extension of M_{22}.

Table 5.7 The strength function $\beta_{22}(x_1,0)$ of the boundary layer for the clamped plate.

x_1	3 DIM	(1,1,2)
0.4990	15.43	13.86
0.4996	18.17	13.41
0.4999	23.85	13.32
0.49993	24.13	13.32
0.49999	24.78	13.30

We see that the 3 dimensional solution shows very strong boundary layer for the moment M_{22}. The model (1.1.2) shows a relatively strong boundary layer and reasonable accuracy, but RM model leads to very poor results in the boundary area.

Let us mention (see [35]) that theoretical strength of the boundary layer for the model (1,1,2) is $\sqrt{\frac{120}{1-\nu}} = 13.1$ which is in very good agreement with the data in Table 5.7. For additional details see also [36].

Summarizing we see that no model guarantees good results and hence only adaptive approaches with a-posteriori estimates could lead to the accuracy with prescribed accuracy.

6) Solution behavior in the neighborhood of the corners

The solution of the three dimensional plate formulation has singular behavior in the neighborhood of the edges and vertices and the exact character of this singular behavior is known [37] [38]. Although the stresses are in general singular in the neighborhood of all edges, the moments and shear forces computed from the three dimensional solutions are singular only in the neighborhood of the corners of the boundary Γ of ω. This behavior is also influenced by the boundary layer. Nevertheless in the very small neighborhood of the corner of Γ (for simplicity placed in the origin) the moments M_{ij} and shear forces Q_{3j} can be written (for $r \ll d$) in the form

$$M_{ij} = C_{ij} r^{\lambda_1 - 1} \varphi_{ij}(\theta) + \text{smoother terms}$$

(6.1)

$$Q_{3j} = D_j r^{\lambda_2 - 1} \psi_j(\theta) + \text{smoother terms}$$

where r, θ are polar coordinates and C_{ij}, D_j are analogs to the stress intensity factors of the plane elasticity.

Any model, for example RM, n-model or t ` Kirchhoff model (which will be addressed in Section 7), yields the solution behavior of the same form as in (6.1) but the coefficient λ_i, $i = 1,2$, and the functions $\varphi_{i,j}(\theta)$, $\psi_j(\theta)$ depend on the model.

For the analysis and the determination of λ_i we refer to [26] where we have proven the following theorem.

<u>Theorem 5.1</u>   i) For any model (n,n,n+1), $n \geq 1$ the coefficients λ_i in (6.1) are the same as for the 3 dimensional solution.

ii) The Riessner-Mindlin, as well as the Kirchhoff model, yield in general the singular behavior which is different when compared with the 3 dimensional solution.

In [26] (see also [36]), we address these singular behaviors in details. Here we will illustrate the results on only one example.

Consider the square plate uniformly bounded, which is clamped on the bounded sides Γ_2, Γ_4 and free on the vertical sides Γ_1, Γ_3 (see (2.2)).

The singularity coefficients λ_1, λ_2 for the Riessner-Mindlin, (1,1,2), and Kirchhoff model are given in Table 6.1.

Table 6.1   The singularity coefficients λ_1, λ_2 for various models

	RM	3 DIM (1,1,2)	K
λ_1	0.7583	0.7112	1.0687 +i 0.4386
λ_2	1	1	0.687 +i 0.4386

In the case when λ_1 in (6.1) is an integer, the notion of smoother terms has to be properly understood. Nevertheless, we will not go here in details, although this case is present in our example. If the coefficient λ is a complex, then (6.1) has to be understood as the real part of this expression and the moments and shear forces oscilate.

In Fig. 6.1, we show in log-log scale, the moment M_{22} on the line $x_1 = x_2$ as the function of r (distance from the vertex of the plate). We used d = 0.01 and $\nu = 0.3$. In the figure we also show the theoretical slope based on the data shown in the table 6.1.
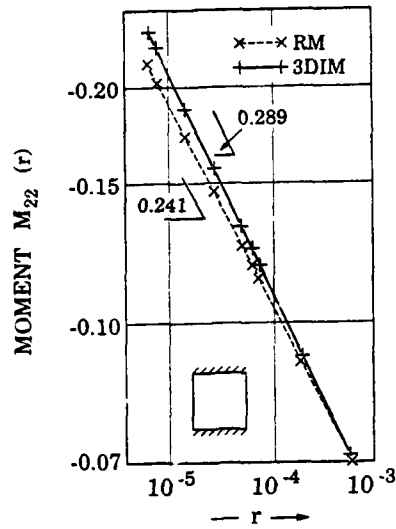
Figure 6.1 The moment M_{22} for the 3 dimensional solution and RM model.

In Figure 6.2 we show the moments M_{11}, M_{22} as well shear force Q_{32} for the RM model.
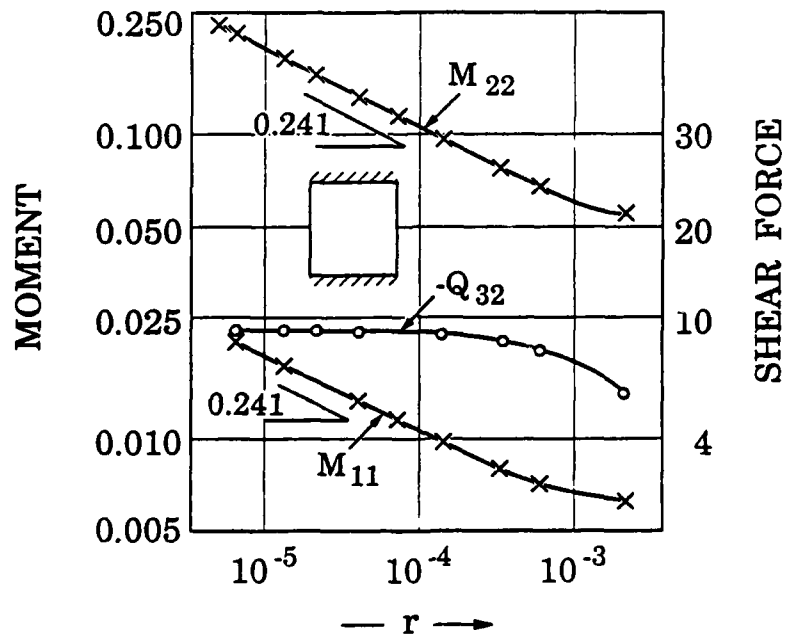

Figure 6.2 The behavior of M_{11}, M_{22} and Q_{32} for the RM model.

Returning to the table 6.1 we see that the Kirchhoff model yields oscilating behavior of the moment M_{22}. We also mentioned that for $d \longrightarrow 0$, all the models converge to the Kirchhoff solution *inside* the domain ω. Hence we can expect that on the line $x_1 = x_2$ we will see increased number of oscillations in the moment M_{22} as $d \longrightarrow 0$. Figure 6.3 shows the behavior of the moment M_{22} computed from 3 dimensional solution for $d = 0.0.1$. We see clearly the effect of the predicted oscillation although in a weak form.
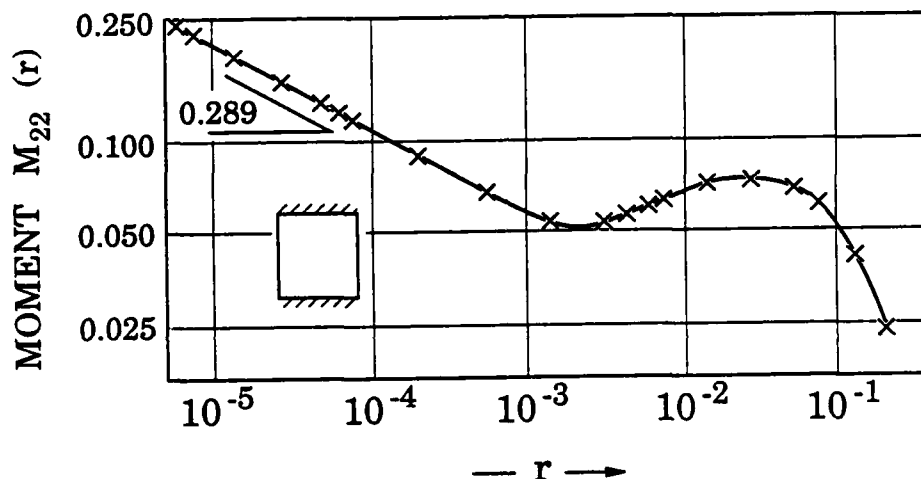
Figure 6.3 The behavior of the moment M_{22} for the 3 dimensional solution.

7) The Kirchhoff model.

The Kirchhoff model describes the limiting solution of the plate problem as $d \longrightarrow 0$. It can describe only the restricted set of the boundary conditions. Let us address this condition in relation to the problem of simply supported plate and one would like to know whether the Kirchhoff model approximates the hard or soft simple support. Because of physical interpretation of the Kirchhoff hypotheses, we can expect that it approximates the <u>hard</u> simple support. This can be further supported by the following observations.

The Kirchhoff model leads to the polygon paradox as the hard simple support in 3 dimensional setting does.

Table 7.1 shows $(d = 0.01, \nu = 0.3)$ the comparison between the Kirchhoff and 3 dimensional formulation

Table 7.1 The comparison of the Kirchhoff and 3 dimensional formulation.

	$Q_{31}(0.4, 0.5)$	$Q_{31}(0.5-0, 0.5)$	$M_{12}(0.5-0, 0.5)$
Hard support	0	0	0.0325
Soft support	-9.72	-6.79	0
Kirchhoff	0	0	0.0325

As usually we denoted

$$Q_{31}(0.5-0,\ 0.5) = \lim_{x \to 0.5} Q_{31}(x,\ 0.5)$$

Often in the Kirchhoff theory (see [39]), the reaction, for example $V_2 = Q_{32}$ (for $x_2 = 0.5$), is computed from the formula

$$V_2 = \left[Q_{32} - \frac{\partial M_{12}}{\partial x_1} \right].$$

The formula tries to simulate the reaction for the soft support by the solution for the hard support. Although this simulation is partly successful, it does not lead to the limiting value for $d \longrightarrow 0$. Further the approximation values for $Q_{32}(x_1, 0.5)$ cannot be predicted at all (see Table 7.1).

The Kirchhoff model in the case of simple support, leads to the negative concentrated reaction in the corner. It is interesting to compare the value of this reaction with the integral of negative shear force (reaction for the soft support. We get $R = 0.0290$ compared with the Kirchhoff value $V = 0.0325$, i.e. 10% difference. For more details we refer to [35].

8. The Optimal Models

In the Section 3, we assumed the form (3.1) as the base for the approximate solution of the plate problem. In the Section 4 we concluded that, from the asymptotic point of view, $\varphi_{i,j}(\eta)$ should be a polynomials. We can ask the question whether polynomials are optimal *when* d *is fixed*. The functions $\varphi_{i,j}^0(\eta)$ will be optimal if they give the minimal error among all possible choices of $\varphi_{i,j}(\eta)$ (e.g. in the energy norm). In [31] it has been shown that polynomials are not optimal for $d > 0$. The optimal $\varphi_{i,j}(\eta)$ are essentially of the form $\sh \alpha \eta$ where α depend on the class of function under consideration and the thickness d. The proper choices are of great importance for the laminated plates. For more details about the hierarchical models for laminated plates we refer to [40].

9. Adaptive modelling and a-posteriori error estimates

We based our models on the form (4.1). Obviously the term $\left(\frac{2x_3}{d}\right)^j$ can be replaced by an arbitrary polynomial of degree $\leq j$ in x_3. We would like to have only a minimal number of terms which will contribute to the solution i.e. that $|u_{i,j}|$ will decrease with increasing j. This can be achieved by selecting $\varphi_{i,j} = P_j\left(\frac{2x_3}{d}\right)$ where P_j is the Legendre polynomial. Then the adaptive approach, which selects different n in the different regions of the plate, can be governed by the requirement that $|u_{i,n_i}| > \varepsilon$, $i = 1,2,3$ in the region, where it is defined. The a-posteriori error estimation can also be extracted from the analysis of the change when n_i is increased. This is in the same spirit as the error estimation in the finite element method. In fact we can understand the modelling as special case of the finite element method [33]. Nevertheless we will not go here in more details. For some results about adaptive approach we refer also to [41].

10. Numerical solution of the plate models.

Using the form (4.1) in the variational principle we can understand it as the p-version of the finite element solution when degree q is used in x_3 variable [33]. In x_1, x_2 the domain ω is partitioned into elements and the h or p or hp version of finite element can be used.

The discretions in the variables x_1, x_2 leads to a *locking problem*. For general theory of locking and quantitative assessment of the locking we refer to [42]. The locking expresses the fact that as $d \longrightarrow 0$, the performance deteriorates. Nevertheless it is necessary to realize that, as $d \longrightarrow 0$, the boundary layer becomes stronger and hence it also influences the performance. Hence in the locking analysis and the numerical experimentation the smoothness of the exact solution has to be independent of d.

Let us address the solution of the RM model (the (1,1,0) model with $B = R$). We assume that in the variables (x_1, x_2) the elements are rectangular of degree p (the serendipity elements) on square mesh of the size h. Then the rate of convergence due to locking is h^{p-1} (measured in the energy norm), while the best possible approximation is of the order h^p. Hence we have a loss of "one unit" in the rate. Nevertheless, this "loss" for higher order elements is visible only for d very small. Let us show an example. We consider the unit square plate Ω_s with hard simple support (because the boundary layer is very week ($\nu = 0.3$)). Figure 10.1 shows the accuracy of the method when uniform mesh of different sizes h is used. The rate h^p is also shown in the Figure 10.1. We see that for $d = 0.01$ and $p = 1$ convergence is virtually not present while for $p = 4$ we do not see any loss of convergence rate for all $0.1 \leq d \leq 0.01$.
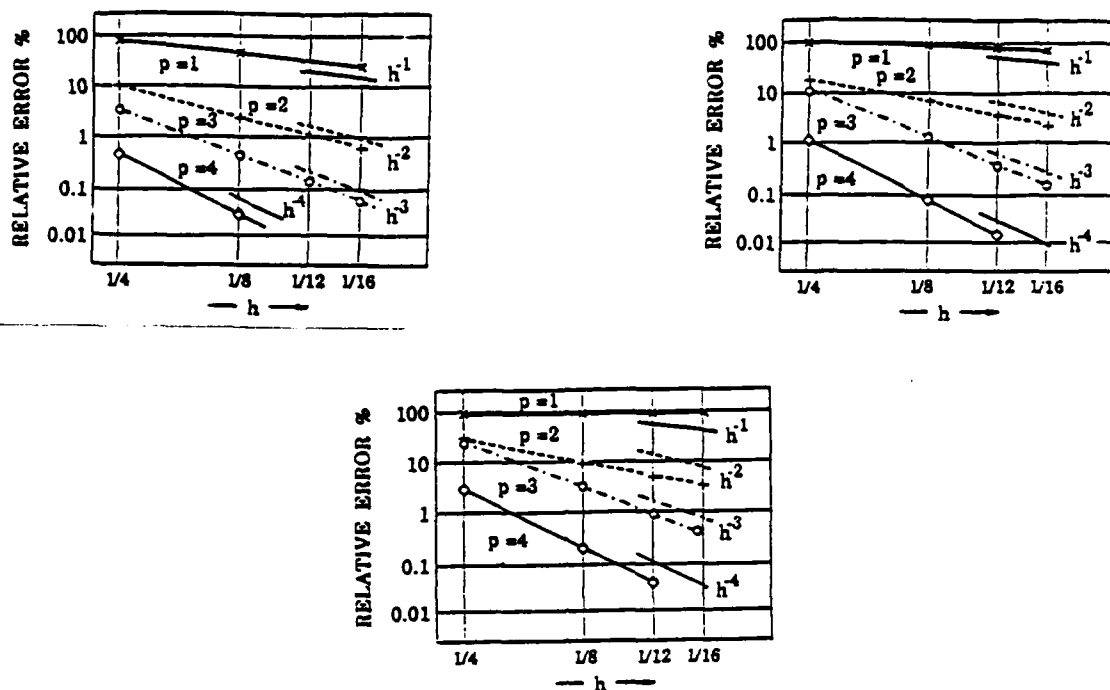
Fig. 10.1   The Convergence of h-version of the RM model a: d = 0.1
b: d = 0.025   c: d = 0.01
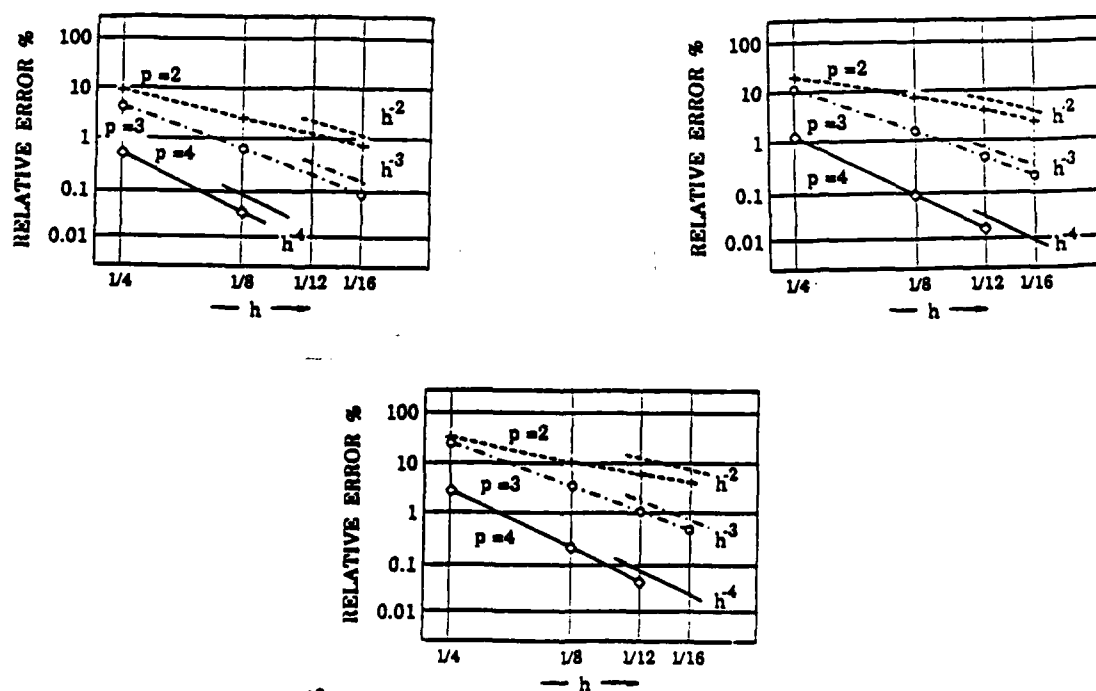
Using the model (1,1,2) we get quite analogous results

Fig. 10.2   The Convergence for the (1,1,2)   model a: d = 0.1
b: d = 0.025   c: d = 0.01

27

Hence the locking is essentially not influenced by the selection of the model.

It is necessarily to realize that higher models show stronger boundary layer and its relation to the mesh size can lead to the wrong conclusion that the locking disappears as $d \longrightarrow 0$. Figure 10.3 shows the effect for the model (3,3,4). For more se [33].

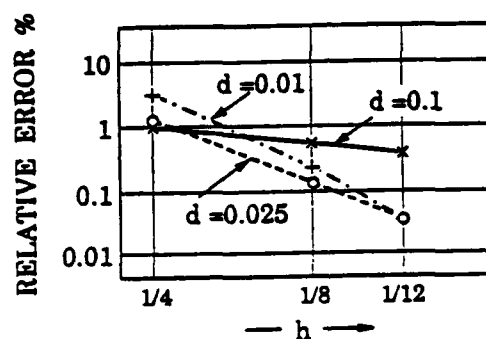Figure 10.3 The convergence of the h-version for the (3,3,4) model (p=4).

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